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CONTROLLABILITY AND STABILIZATION OF PROGRAMMED MOTIONS OF REVERSIBLE MECHANICAL AND ELECTROMECHANICAL SYSTEMS†

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Problems of controllability and methods of stabilizing programmed motions of a large class of mechanical and electromechanical systems which are reversible with respect to the control are considered. Criteria of the controllability and stabilizability of reversible systems are obtained. Programmed motions and algorithms of programmed control are designed in analytical form and algorithms of programmed motions for non-linear reversible systems are synthesized.

1. STATEMENT OF THE PROBLEM

THE DYNAMICS of a large class of mechanical and electromechanical systems (MSs and EMSs) are described by a differential equation of the form

$$z^{\cdot} = F(z, u, t), \quad z(t_0) = z_0, \quad t \geqslant t_0 \tag{1.1}$$

where z_0 and $z = z(t)$ are the n-dimensional vectors of states of the system at the initial and current instants of time, u is the m-dimensional vector of controls, and F is an n-dimensional vector-function satisfying (for a feasible control) the conditions of the existence and uniqueness of a solution of system (1.1) and defining the properties of a specified controlled system (CS).

If the controlled systems are MSs (slave mechanisms of robot-manipulators, lathes, coordinate instrument tools, etc.), their dynamics are described by Lagrange's equations of the second kind in the following form [1–6]

$$\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot}} - \frac{\partial T}{\partial q} + \frac{\partial \Pi}{\partial q} - Q_0 = u, \quad q(t_0) = q_0, \quad q^{\cdot}(t_0) = q^{\cdot}_0, \quad t \geqslant t_0$$

Here q is the m-dimensional vector of generalized coordinates of the MS, $T = \frac{1}{2} q^{\cdot *} A_0(q) q^{\cdot}$ is the kinetic energy of the MS, $A_0(q)$ is an $m \times m$ matrix, $\Pi = \Pi(q)$ is the potential energy of the MS, u is the m-dimensional vector of the controls, and the asterisk denotes transposition.

In this case, Eq. (1.1) has the order $n = 2m$, and

$$z = \begin{Vmatrix} q \\ q^{\cdot} \end{Vmatrix}, \quad F(z, u, t) = \begin{Vmatrix} q^{\cdot} \\ A_0^{-1}(q)(-b_0(q, q^{\cdot}, t) + u) \end{Vmatrix} \tag{1.2}$$

For EMSs containing DC motors with rigid reduction gears, Eq. (1.1) has the order $n = 3m$, and [5]

$$z = \begin{Vmatrix} q \\ q^{\cdot} \\ I \end{Vmatrix}, \quad F(z, u, t) = \begin{Vmatrix} q^{\cdot} \\ A^{-1}(q)(k_m I - b(q, q^{\cdot}, t)) \\ L^{-1}(u - RI - k_e i_p q^{\cdot}) \end{Vmatrix} \tag{1.3}$$

$$A(q) = J i_p + i_p^{-1} A_0(q), \quad b(q, q^{\cdot}, t) = k_0 i_p q^{\cdot} + i_p^{-1} b_0(q, q^{\cdot}, t)$$

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Here I is the m-dimensional vector of the currents in the armature circuits, u is the m-dimensional vector of the controlling voltages applied to the armature circuits of the motor, J, k_0, k_m, L, R and k_e are the diagonal matrices of electromechanical parameters of the motor which are positive quantities, i_p is the diagonal matrix of the ratios of the reduction gears (such that $\varphi = i_p q$ where φ is the vector of the angles of rotation of the motor shafts), and $A(q)$ is a non-degenerate matrix.

System (1.1) is said to be controllable if for any two states $z_{p0} \in R^n$ and $z_{p1} \in R^n$ (where R^n is n-dimensional Euclidean space) and for arbitrary values of $t_0 < t_1 < \infty$ a control $u(t)$ exists that the corresponding solution $z(t)$ of system (1.1) satisfies the boundary conditions

$$z(t_0) = z_{p0}, \quad z(t_1) = z_{p1} \tag{1.4}$$

The solution $z = z_p(t)$ of system (1.1) which satisfies boundary conditions (1.4) will be called programmed motion, and the control

$$u = u_p(t), \quad t \in [t_0, t_1], \quad (t_1 \quad t_0 < \infty) \tag{1.5}$$

corresponding to it will be called the programmed control.

Let us consider a programmed motion $z_p = z_p(t)$, $t \geqslant t_0$ of system (1.1). We will say that it is stabilizable if a control with feedback in the state vector in the form

$$u = u(z, t), \quad t \geqslant t_0 \tag{1.6}$$

exists which ensures asymptotic stability of the programmed motion $z_p(t)$.

The problems under consideration in this paper concern the analysis of the conditions of controllability and stabilizability of non-linear MSs and EMSs. Algorithms for designing programmed motions and for stabilizing them are synthesized.

The methods proposed for solving the above problems generalize and develop the results obtained in [1–11]. They are based on the property of the reversibility of the dynamical equations of MSs and EMSs with respect to the control.

2. THE REVERSIBILITY OF THE DYNAMICAL EQUATIONS AND THE TRANSFORMATION OF THE COORDINATES TO CANONICAL FORM

The structure of the dynamical equations of MSs (1.1) and (1.2) and EMSs (1.1) and (1.3) is such that they may be represented in the form of system (1.1), where

$$z = \operatorname{col}(z_1, \ldots, z_r), \quad n = mr \tag{2.1}$$

$$F(z, u, t) = \operatorname{col}(F_1(z^2, t), \ldots, F_{r-1}(z^r, t), F_r(z, u, t)) \tag{2.2}$$

$$F_i(z^{i+1}, t) = C_i(z^i, t) + D_i(z^i, t) z_{i+1}, \quad i = 1, \ldots, r-1 \tag{2.3}$$

$$F_r(z, u, t) = z_r^{\cdot} = C_r(z, t) + D_r(z, t) u \tag{2.4}$$

Here z_i is an m-dimensional vector, $z^i = \operatorname{col}(z_1, \ldots, z_i)$ are mi-dimensional vectors, and C_i and D_i $(i = 1, \ldots, r)$ are specified vector functions and matrix functions.

Henceforth, we assume that vector functions F_i (2.3) and (2.4) $(i = 1, \ldots, r)$ are continuously differentiable a sufficient number of times with respect to their arguments.

For MSs (1.1) and (1.2) we have

$$\begin{gathered} z = \operatorname{col}(q, q^{\cdot}), \quad n = 2m, \quad F_1(z^2, t) = q^{\cdot}, \quad F_2(z, u, t) = A_0^{-1}(q)(-b_0(q, q^{\cdot}, t) + u), \\ C_1(z^1, t) = 0, \quad D_1(z^1, t) = I_m, \\ C_2(z^2, t) = -A_0^{-1}(q) b_0(q, q^{\cdot}, t), \quad D_2(z^2, t) = A_0^{-1}(q) \end{gathered} \tag{2.5}$$

(I_m is the unit $m \times m$ matrix). For EMSs (1.1) and (1.3) we have

$$\begin{gathered} z = \operatorname{col}(q, q^{\cdot}, I), \quad n = 3m, \quad F_1(z^2, t) = q^{\cdot}, \\ F_2(z^3, t) = A^{-1}(q)(k_m I - b(q, q^{\cdot}, t)), \quad F_3(z, u, t) = L^{-1}(u - RI - k_e i_p q^{\cdot}) \\ C_1(z^1, t) = 0, \quad D_1(z^1, t) = I_m, \quad C_2(z^2, t) = -A^{-1}(q) b(q, q^{\cdot}, t) \\ D_2(z^2, t) = A^{-1}(q) k_m, \quad C_3(z^3, t) = -L^{-1}(RI + k_e i_p q^{\cdot}), \quad D_3(z^3, t) = L^{-1} \end{gathered} \tag{2.6}$$

The matrices D_i $(i = 1, \ldots, r)$ are non-degenerate for MSs and EMSs. Hence, the dynamical equations (1.1), (2.1)–(2.4) may be written in the form

$$z_i^{\cdot} = F_i(z^{i+1}, t) = C_i(z^i, t) + D_i(z^i, t) z_{i+1}, \quad i = 1, \ldots, r-1 \tag{2.7}$$

$$u = D_r^{-1}(z, t)(z_r^{\cdot} - C_r(z, t)) \tag{2.8}$$

which are added for the control. The MSs and EMSs having this property will be called reversible controlled systems (RCS).

It is convenient to synthesize the stabilizing control laws using the canonical variables in the form [1–8]

$$x = \operatorname{col}(x_1, \ldots, x_r), \quad x_1 = z_1, \quad x_i = x_{i-1}^{\cdot}, \quad i = 2, \ldots, r \tag{2.9}$$

instead of the original "physical" coordinates (2.1).

Let us find the coordinate transformations relating x and z. In the general case these one-to-one transformations have the form

$$x = \Psi(z, t), \quad z = \Phi(x, t) \tag{2.10}$$

The vector functions $\Psi(z, t)$ and $\Phi(x, t)$ are determined in Appendix 1.

In the case of MSs (1.1), (2.2)–(2.5), we have

$$\Psi(z, t) = \Phi(x, t) = \begin{Vmatrix} q \\ q^{\cdot} \end{Vmatrix} \tag{2.11}$$

For EMSs (1.1), (2.2)–(2.4), (2.6) we have

$$\Psi(z, t) = \begin{Vmatrix} q \\ q^{\cdot} \\ A^{-1}(q)(k_m I - b(q, q^{\cdot}, t)) \end{Vmatrix}, \quad \Phi(x, t) = \begin{Vmatrix} q \\ q^{\cdot} \\ k_m^{-1}(A(q) q^{\cdot\cdot} + b(q, q^{\cdot}, t)) \end{Vmatrix} \tag{2.12}$$

We write RCS (1.1), (2.1)–(2.4) using the canonical variables

$$x^{\cdot} = Px + Q(R(x, t) + S(x, t) u) \tag{2.13}$$

Here $R(x, t)$ and $S(x, t)$ are the m-vector and $m \times m$ matrix, respectively, specified in Appendix 2, P and Q are the $n \times n$ and $n \times m$ matrices

$$P = \begin{Vmatrix} 0 & \mathrm{I}_{n-m} \\ 0 & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 \\ \mathrm{I}_m \end{Vmatrix} \tag{2.14}$$

and O is the zero matrix of corresponding dimensions.

In the case of MSs we have $n = 2m$, and

$$P = \begin{Vmatrix} 0 & \mathrm{I}_m \\ 0 & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 \\ \mathrm{I}_m \end{Vmatrix} \tag{2.15}$$

$$R(x, t) = -A_0^{-1}(q) b_0(q, q^{\cdot}, t), \quad S(x, t) = A_0^{-1}(q)$$

For EMSs we have $n = 3m$, and

$$P = \begin{Vmatrix} 0 & \mathrm{I}_{2m} \\ 0 & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 \\ \mathrm{I}_m \end{Vmatrix} \tag{2.16}$$

$$R(x, t) = -A_1^{-1}(q) b_1(q, q^{\cdot}, q^{\cdot\cdot}, t), \quad S(x, t) = A_1^{-1}(q)$$

$$A_1(q) = Lk_m^{-1} A(q), \quad b_1(q, q^{\cdot}, q^{\cdot\cdot}, t) = (Lk_m^{-1} A^{\cdot}(q) + Rk_m^{-1} A(q)) q^{\cdot\cdot} +$$
$$+ Lk_m^{-1} b^{\cdot}(q, q^{\cdot}, t) + Rk_m^{-1} b(q, q^{\cdot}, t) + k_e i_p q^{\cdot}$$

$A_1(q)$ is the non-degenerate matrix.

Because the matrix $S(x, t)$ in (2.13) is non-degenerate, Eq. (2.13) can be solved for the control in the form

$$u = S^{-1}(x, t)(x_r^{\cdot} - R(x, t)) \tag{2.17}$$

Hence, the dynamical equations of MS (1.1), (2.1)–(2.5) and EMS (1.1), (2.1)–(2.4), (2.6) can be represented in canonical form (2.13) which can be solved for the control in form (2.17).

3. CONTROLLABILITY AND ALGORITHMS FOR DESIGNING PROGRAMMED CONTROLS AND PROGRAMMED MOTIONS

First, we will demonstrate that system (2.13) is controllable.

Consider the auxiliary control

$$w = R(x, t) + S(x, t)u \tag{3.1}$$

System (2.13) then takes the form

$$\dot{x} = Px + Qw \tag{3.2}$$

It can be shown that

$$\operatorname{rank}[Q, PQ, \ldots, P^{r-1}Q] = n$$

Hence [12], system (3.2) is controllable. This means that the control law $w = w_p(t) = w_p$ exists transferring (3.2) from any initial state $x_p(t_0) = x_{p0}$ to any final state along the trajectory $x = x_p(t) = x_p$ in a time $t_1 - t_0 < \infty$.

Using (3.1) we obtain

$$u = S^{-1}(x, t)(w - R(x, t)) \tag{3.3}$$

If we substitute the control law $w = w_p(t)$, $x = x_p(t)$ into (3.3) we obtain the control law

$$u = u_p = u_p(t) = S^{-1}(x_p, t)(w_p - R(x_p, t)) \tag{3.4}$$

which ensures that system (2.13) is transferred from any initial state $x_p(t_0) = x_{p0}$ to any final state $x_p(t_1) = x_{p1}$ along the trajectory $x = x_{p1}$ in a time $t_1 - t_0 < \infty$.

It follows from the controllability of canonical system (2.13) and the coordinate transformation (2.10) that the control law

$$u = S^{-1}(\Psi(z, t), t)(w - R(\Psi(z, t), t)) \tag{3.5}$$

where $w = w_p(t)$, $z = z_p(t) = \Phi(x_p, t)$, transfers the original system (1.1), (2.1)–(2.4) from the initial state $z_{p0} = \Phi(x_{p0}, t_0)$ to the final state $z_{p1} = \Phi(x_{p1}, t_1)$ along the trajectory

$$z = z_p = \Phi(x_p, t) \tag{3.6}$$

in a time $t_1 - t_0 < \infty$.

Hence, the original system (1.1), (2.1)–(2.4) is controllable.

For the system written in canonical form (2.13) and (2.14) the criterion of controllability has the form

$$\operatorname{rank} S(x, t) = m, \quad x \in R^n, \quad t \geqslant t_0 \tag{3.7}$$

and for original system (1.1), (2.1)–(2.4) it implies the condition

$$\operatorname{rank} D_i(z^i, t) = m, \quad z^i \in R^{mi}, \quad t \geqslant t_0, \quad i = 1, \ldots, r \tag{3.8}$$

Note that criteria (3.7) and (2.8) are satisfied for the class of MSs and EMSs under consideration.

Let us design the programmed control and the programmed motion in analytical form. For this purpose, we will first seek an auxiliary programmed control and programmed motion for linear canonical system (3.12) and (3.14), i.e. we construct

$$w = w_p(t), \quad x = x_p(t), \quad t \in [t_0, t_1] \quad (t_1 - t_0 < \infty), \tag{3.9}$$

satisfying the boundary conditions

$$x_p(t_0) = x_{p0}, \quad x_p(t_1) = x_{p1} \tag{3.10}$$

We will seek $w_p(t)$ in the form

$$w_p(t) = Q^* e^{P^*(t_1 - t)} \alpha, \quad t \in [t_0, t_1] \tag{3.11}$$

where α is the required constant n-dimensional vector. At the instant t_1 the solution $x(t)$ of system (3.2), (2.14) and (3.11) takes the form

$$x_{p1} = e^{P(t_1 - t_0)} x_{p0} + K\alpha \tag{3.12}$$

$$K = \int_{t_0}^{t_1} e^{P(t_1 - t)} Q Q^* e^{P^*(t_1 - t)} dt \tag{3.13}$$

Because of the controllability of system (3.2) and (2.14), the matrix K (3.13) is positive definite [13], and hence we obtain from (3.12)

$$\alpha = K^{-1}(x_{p1} - e^{PT} x_{p0}), \quad T = t_1 - t_0 \tag{3.14}$$

Eliminating the vector α (3.14) from (3.11) we find the required auxiliary programmed control in the form

$$w_p(t) = Q^* e^{P^*(t_1 - t)} K^{-1}(x_{p1} - e^{PT} x_{p0}) \tag{3.15}$$

Taking (2.14) into account we have

$$e^{P(t_1 - t)} = \sum_{k=0}^{r-1} \frac{P^k (t_1 - t)^k}{k!}, \quad e^{PT} = \sum_{k=0}^{r-1} \frac{P^k T^k}{k!}$$

$$Q^* \exp[P^*(t_1 - t)] = [((t_1 - t)^{r-1}/(r-1)!) I_m, \ldots, ((t_1 - t)^2/2!) I_m, (t_1 - t) I_m, I_m]$$

$$K_{ij} = (T^{2r-i-j+1}/((2r - i - j + 1)(r - i)!(r - j)!)) I_m, \quad i, j = 1, \ldots, r \tag{3.16}$$

where K_{ij} $(i, j = 1, \ldots, r)$ are $m \times m$ blocks of the matrix K (3.13). Using (3.14)–(3.16) we can represent the auxiliary programmed control in the form

$$w_p(t) = \sum_{k=0}^{r-1} \frac{\beta_k (t_1 - t)^{r-1-k}}{(r - 1 - k)!}, \quad t \in [t_0, t_1] \tag{3.17}$$

where the m-dimensional vectors β_k $(k = 0, \ldots, r-1)$ are such that the equality

$$\beta = \mathrm{col}(\beta_0, \ldots, \beta_{r-1}) = K^{-1}\Big(x_{p1} - \Big(\sum_{k=0}^{r-1} \frac{P^k T^k}{k!}\Big) x_{p0}\Big) \tag{3.18}$$

holds.

Using (3.17), (3.18) and the relations

$$e^{P(t_1 - t_0)} = \sum_{k=0}^{r-1} \frac{P^k (t_1 - t_0)^k}{k!}, \quad e^{P(t-\tau)} = \sum_{k=0}^{r-1} \frac{P^k (t - \tau)^k}{k!}$$

we can construct programmed motion (3.9) and (3.10) which corresponds to programmed control (3.17) and (3.18), in the form

$$x_p(t) = \Big(\sum_{k=0}^{r-1} \frac{P^k (t - t_0)^k}{k!}\Big) x_{p0} + \int_{t_0}^{t} \Big(\sum_{k=0}^{r-1} \frac{P^k (t - \tau)^k}{k!}\Big) Q \times$$

$$\times \Big(\sum_{k=0}^{r-1} \frac{\beta_k (t_1 - \tau)^{r-1-k}}{(r - 1 - k)!}\Big) d\tau, \quad t \in [t_0, t_1] \tag{3.19}$$

If we substitute expressions (3.17)–(3.19) obtained for $w_p(t)$ and $x_p(t)$, respectively, into (3.4), we obtain the required programmed control for the RCS written in canonical variables (2.13) and (2.14). Using $w = w_p(t)$ (3.17), (3.18), $x = x_p(t)$ (3.19) and coordinate transformation (2.10) we obtain programmed control (3.5) and the programmed motion $z_p(t) = \Phi(x_p, t)$, $t \in [t_0, t_1]$ corresponding to it for the original RCS (1.1), (2.1)–(2.4).

4. STABILIZABILITY AND ALGORITHMS FOR THE STABILIZATION OF PROGRAMMED MOTIONS

It follows from the controllability of linear system (3.2) that a constant $m \times m$ matrix Γ_0 exists such that the matrix

$$\Gamma = P + Q\Gamma_0 \tag{4.1}$$

is stable, i.e. we have

$$\operatorname{Re} \lambda_i(\Gamma) < 0, \quad i = 1, \ldots, n \tag{4.2}$$

where $\lambda_i(\Gamma)$ $(i = 1, \ldots, n)$ are the eigenvalues of Γ.

Consider the auxiliary control law with feedback in x

$$w = Q^* \dot{x}_p + \Gamma_0(x - x_p) = Q^*(\dot{x}_p + \Gamma(x - x_p)) = w_p + \Gamma_0(x - x_p) \tag{4.3}$$

The equation of the transient

$$e_x = e_x(t) = x(t) - x_p(t), \quad t \geqslant t_0 \tag{4.4}$$

in closed system (3.2), (4.1)–(4.3) then have the form

$$\dot{e}_x = \Gamma e_x, \quad e_x(t_0) = e_{x_0} = x_0 - x_p(t_0), \quad t \geqslant t_0 \tag{4.5}$$

Hence, the programmed motion $x_p(t)$ of system (3.2), (4.1)–(4.3) is asymptotically stable as a whole. The transient e_x (4.4) of the system satisfies the limit

$$|e_x(t)| \leqslant c_1 \exp(\gamma(t - t_0)) |e_x(t_0)|, \quad t \geqslant t_0 \tag{4.6}$$
$$\gamma = \max_i \operatorname{Re} \lambda_i(\Gamma) \quad (i = 1, \ldots, n)$$

where $c_1 > 0$ is a parameter which depends on Γ only.

If we substitute (4.1)–(4.3) into (3.3) using (3.4), we obtain the stabilizing control law

$$\begin{aligned} u &= S^{-1}(x, t)(Q^* \dot{x}_p + \Gamma_0(x - x_p) - R(x, t)) = \\ &= S^{-1}(x, t)(Q^*(\dot{x}_p + \Gamma(x - x_p)) - R(x, t)) = \\ &= S^{-1}(x, t)(w_p + \Gamma_0(x - x_p) - R(x, t)) = \\ &= S^{-1}(x, t)(S(x_p, t) u_p + R(x_p, t) + \Gamma_0(x - x_p) - R(x, t)) \end{aligned} \tag{4.7}$$

in the canonical variables for RCS (2.13) and (2.14).

The equation of the transient in closed system (2.13), (2.14), (4.7), (4.1) and (4.2) has the form (4.5), (4.1) and (4.2), and consequently the programmed motion $x_p(t)$ is asymptotically stable as a whole, i.e. the programmed motion $x_p(t)$ is stabilizable and estimate (4.6) holds for the transient.

It can be shown that in original RCS (1.1), (2.1)–(2.4) the programmed motion $z_p(t)$ is stabilizable.

If we substitute (2.10) into (4.7), (4.1) and (4.2), we obtain the stabilizing control law in original "physical" coordinates

$$\begin{aligned} u &= S^{-1}(\Psi(z, t), t)(Q^* \dot{\Psi}(z_p, t) + \Gamma_0(\Psi(z, t) - \Psi(z_p, t)) - R(\Psi(z, t), t)) = \\ &= S^{-1}(\Psi(z, t), t)(S(\Psi(z_p, t), t) u_p + R(\Psi(z_p, t), t) + \Gamma_0(\Psi(z, t) - \Psi(z_p, t)) - R(\Psi(z, t), t)) \end{aligned} \tag{4.8}$$

The equation of the transient in closed system (1.1), (2.1)–(2.4), (4.8), (4.1) and (4.2) has the form

$$\begin{aligned} \dot{e}_z = {} & F(e_z + z_p, S^{-1}(\Psi(e_z + z_p, t), t)(S(\Psi(z_p, t), t) u_p + R(\Psi(z_p, t), t) + \Gamma_0(\Psi(e_z + z_p, t) - \Psi(z_p, t)) - \\ & - R(\Psi(e_z + z_p, t), t)), t) - F(z_p, u_p, t), \quad t \geqslant t_0, \quad e_z = z - z_p \end{aligned} \tag{4.9}$$

Let us estimate the transient in (3.15). We will assume that for each of the vector functions C_i ($i = 1, \ldots, r$) and matrix functions D_i ($i = 1, \ldots, r$) in (2.3) and (2.4) for all possible values of the arguments the following limits hold

$$|C_i(z^i, t)| \leq a_{1i} + a_{2i}|z^i|^{ki}, \quad |D_i(z^i, t)| \leq a_{3i}, \quad i = 1, \ldots, r \tag{4.10}$$

Here $a_{1i} \geq 0$, $a_{2i} \geq 0$, $a_{3i} \geq 0$ ($ki \geq 1$) are constants. We will assume that similar limits hold for the partial derivatives of C_i and D_i with respect to their arguments.

Then, using the limits of the final increments of the vector function $\Phi(e_x + x_p, t) - \Phi(x_p, t)$ it can be shown that the transient $e_z(t) = z(t) - z_p(t)$ satisfies the limit

$$|e_z(t)| = |z(t) - z_p(t)| = |\Phi(e_x(t) + x_p(t), t) - \Phi(x_p(t), t)| =$$

$$= |\int_0^1 \partial(\Phi(\nu e_x(t) + x_p(t), t)) - \Phi(x_p(t), t)/\partial e_x(t) d\nu e_x(t)| \leq$$

$$\leq |\int_0^1 \partial(\Phi(\nu e_x(t) + x_p(t), t - \Phi(x_p(t), t))/\partial e_x(t)\, d\nu||e_x(t)| \leq$$

$$\leq \mu_0(t)|e_x(t)| \leq \mu(t) c_1 \exp(\gamma(t - t_0))|e_x(t_0)| \leq$$

$$\leq \mu(t)\exp(\gamma(t - t_0))|\Psi(z(t_0), t_0) - \Psi(z_p(t_0), t_0)|, \quad t \geq t_0 \tag{4.11}$$

where $\mu_0(t) = a_1 + a_2|e_x^{r-1}(t)|^s$, $\mu(t) = \mu_0(t)c_1$, $a_1 > 0$, $a_2 \geq 0$, $s \geq 1$ are constants and $e_x^{r-1}(t) = x^{r-1}(t) - x_p^{r-1}(t)$.

From (4.11), (4.6) and (4.2) it follows that the programmed motion $z_p(t)$ in the original system (2.1), (1.1) and (2.4) with control law (4.8), (4.1) and (4.2) is asymptotically stable as a whole, i.e. the programmed motion $z_p(t)$ is stabilizable.

Thus, the control laws (4.7), (4.1), (4.2) and (4.8), (4.1), (4.2) synthesized above ensure asymptotic stability as a whole of the programmed motions $x_p(t)$ and $z_p(t)$ for corresponding RCSs (2.13), (2.14) and (1.1), (2.1)–(2.4) with limits (4.6) and (4.11) for transients.

By specifying the matrix Γ_0 of gains it is possible to obtain the desired damping of the transients. For instance, to obtain an aperiodic transient in closed systems it is sufficient to require that the spectrum of matrix Γ (4.1) should consist of real negative numbers only.

In the case that the matrix Γ_0 in (4.1) consists of diagonal $m \times m$ blocks $\Gamma_{0k} = \mathrm{diag}(\Gamma_{0ki})_{i=1}^m$ ($k = 0, \ldots, r-1$), so that $\Gamma_0 = [-\Gamma_{00}, \ldots, -\Gamma_{0,r-1}]$, it is possible to decompose the equation of the transient in the closed system (which is an RCS described in canonical form (2.13) and (2.14) with control law (4.7), (4.1) and (4.2)) into m independent equations of the rth order of the form

$$e_{x_{1i}}^{(r)} + \Gamma_{0,r-1,i} e_{x_{1i}}^{(r-1)} + \ldots + \Gamma_{00i} e_{x_{1i}} = 0, \quad i = 1, \ldots, m$$

Here $e_{x_{1i}}^{(k)}$ is the kth derivative of the coordinate $e_{x_{1i}} = e_x^{(0)} = x_{1i} - x_{1ip}$ with respect to time; the transient e_x (4.4) may be presented in the form $e_x = \mathrm{col}(e_{x_1}, \ldots, e_{x_r})$, $e_{x_1} = \mathrm{col}(e_{x_{11}}, \ldots, e_{x_{1m}}) = x_1 - x_{1p}$, $e_{x_i} = e^{\bullet}_{x_{i+1}}$, $i = 2, \ldots, r$.

APPENDIX 1

Taking into account relations (2.9) and (2.3) and the fact that the matrices D_i ($i = 1, \ldots, r-1$) are non-degenerate it is possible to obtain coordinate transformations (2.10) relating x and z. We have

$$x = \Psi(z, t) = \mathrm{col}(\Psi_1(z^1, t), \ldots, \Psi_r(z^r, t)) \tag{A1.1}$$

Here

$$\begin{aligned}
&\Psi_1(z^1, t) = x_1 = z_1 = z_1^1 = K_1 + L_1 z_1 \\
&\Psi_2(z^2, t) = x_2 = x_1^{\cdot} = \Psi_1^{\cdot}(z^1, t) = z_1^{\cdot} = F_1(z^2, t) = \\
&= C_1(z^1, t) + D_1(z^1, t) z_2 = K_2(z^1, t) + L_2(z^1, t) z_2 \\
&\Psi_i(z^i, t) = x_i = x_{i-1}^{\cdot} = \Psi_{i-1}^{\cdot}(z^{i-1}, t) = K_{i-1}^{\cdot}(z^{i-2}, t) + (L_{i-1}(z^{i-2}, t) z_{i-1}) = \\
&= K_{i-1}^{\cdot}(z^{i-2}, t) + L_{i-1}^{\cdot}(z^{i-2}, t) z_{i-1} + L_{i-1}(z^{i-2}, t) F_{i-1}(z^i, t) = \\
&= K_{i-1}^{\cdot}(z^{i-2}, t) + L_{i-1}^{\cdot}(z^{i-2}, t) z_{i-1} + L_{i-1}(z^{i-2}, t)(C_{i-1}(z^{i-1}, t) + \\
&+ D_{i-1}(z^{i-1}, t) z_i) = K_i(z^{i-1}, t) + L_i(z^{i-1}, t) z_i, \quad i = 3, \ldots, n
\end{aligned} \tag{A1.2}$$

where K_i ($i = 1, \ldots, r$) are vector functions and L_i ($i = 1, \ldots, r$) are matrix functions of the form

$$K_1=0,\quad L_1=I_m,\quad K_2(z^1,t)=C_1(z^1,t),\quad L_2(z^1,t)=L_1D_1(z^1,t)$$
$$K_i(z^{i-1},t)=K_{i-1}^{\cdot}(z^{i-2},t)+L_{i-1}^{\cdot}(z^{i-2},t)z_{i-1}+L_{i-1}(z^{i-2},t)C_{i-1}(z^{i-1},t) \qquad (A1.3)$$
$$L_i(z^{i-1},t)=L_{i-1}(z^{i-2},t)D_{i-1}(z^{i-1},t),\quad i=3,\ldots,r$$

Since the matrices L_i are non-degenerate we obtain the transformation

$$z=\Phi(x,t)=\mathrm{col}(\Phi_1(x^1,t),\ldots,\Phi_r(x^r,t)) \qquad (A1.4)$$

by solving Eq. (A1.2) for Z_i ($i=1,\ldots,r$).

Here we have

$$\Phi_1(x^1,t)=M_1+N_1x_1,\quad \Phi_i(x^i,t)=M_i(x^{i-1},t)+N_i(x^{i-1},t)x_i$$
$$i=2,\ldots,r;\quad x^i=\mathrm{col}(x_1,\ldots,x_i),\quad x^r=x$$
$$M_1=0,\quad N_1=I_m,\quad M_i(x^{i-1},t)=-L_i^{-1}(\Phi^{i-1}(x^{i-1},t),t)K_i(\Phi^{i-1}(x^{i-1},t),t),$$
$$N_i(x^{i-1},t)=L_i^{-1}(\Phi^{i-1}(x^{i-1},t),t) \qquad (A1.5)$$
$$\Phi^{i-1}(x^{i-1},t)=\mathrm{col}(\Phi_1(x^1,t),\ldots,\Phi_{i-1}(x^{i-1},t))$$

APPENDIX 2

The vector function $R(x,t)$ and the matrix function $S(x,t)$ in (2.13) are determined from the expression for $\dot{x}_r$ if relations (2.10), (A1.1)–(A1.5) are taken into account. In fact, we have

$$x_r^{\cdot}=\Psi_r^{\cdot}(z^r,t)=K_r^{\cdot}(z^{r-1},t)+(L_r(z^{r-1},t)z_r)^{\cdot}=$$
$$=K_r^{\cdot}(z^{r-1},t)+L_r^{\cdot}(z^{r-1},t)z_r+L_r(z^{r-1},t)z_r^{\cdot}=$$
$$=K_r^{\cdot}(z^{r-1},t)+L_r^{\cdot}(z^{r-1},t)z_r+L_r(z^{r-1},t)(C_r(z,t)+D_r(z,t)u)=$$
$$=R(x,t)+S(x,t)u \qquad (A2.1)$$

Here

$$R(x,t)=K_r^{\cdot}(\Phi^{r-1}(x^{r-1},t),t)+L_r^{\cdot}(\Phi^{r-1}(x^{r-1},t),t)\Phi_r(x^r,t)+$$
$$+L_r(\Phi^{r-1}(x^{r-1},t),t)C_r(\Phi(x,t),t) \qquad (A2.2)$$
$$S(x,t)=L_r(\Phi^{r-1}(x^{r-1},t),t)D_r(\Phi(x,t),t)$$

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